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2002 J. Phys. A: Math. Gen. 35 6963

(http://iopscience.iop.org/0305-4470/35/32/315)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6963–6972 PII: S0305-4470(02)35314-9

Darboux transformations, infinitesimal symmetries and conservation laws for the nonlocal two-dimensional Toda lattice

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Received 28 March 2002, in final form 12 June 2002
Published 2 August 2002
Online at stacks.iop.org/JPhysA/35/6963

Abstract
The technique of Darboux transformation is applied to the nonlocal partner of the two-dimensional periodic A_{n-1} Toda lattice. This system is shown to admit a representation as the compatibility conditions of direct and dual overdetermined linear systems with a quantized spectral parameter. We give the generalization of the Darboux transformation technique on linear equations of this type. We present the connections between the solutions of overdetermined linear systems and their expansions in series in the neighbourhood of singular points. The solutions of the nonlocal Toda lattice and infinite hierarchies of the infinitesimal symmetries and conservation laws are obtained.

PACS numbers: 05.45.Yv, 02.30.Jr

1. Introduction

It is well known that some nonlocal differential equations of great physical significance (e.g. the Benjamin–Ono (BO) equation [1–3], the intermediate long wave (ILW) equation [4, 5]) possess the same mathematical properties as local nonlinear equations integrable in the framework of the inverse scattering transformation (IST) method [6]. Such nonlocal equations have been found to have the following characteristics: multi-soliton solutions [4, 7–11]; infinitely many conservation laws [12, 15–18]; the Bäcklund transformation [15–18]; to pass the Painlev'e test [18]; to be represented in bilinear form [8–11, 15–18] and as the compatibility condition of the overdetermined linear system (Lax pair) [13–18]; to possess algebro–geometric solutions [19]. An effective technique of Darboux transformation (DT) [20] has also been applied to nonlocal equations, such as the ILW equation, the nonlocal analogue of the Kadomtsev–Petviashvili equation [14, 19] and the nonlocal Toda equation [21]. This technique allows us to obtain the infinite hierarchies of solutions of Lax pairs and associated nonlinear equations arising from the compatibility condition. Carrying out the proper sequence of the DTs adds the one-soliton component of the solution to the initial solution of nonlinear integrable equations.

The noncommutative ('quantum') generalization of the spectral parameter of the usual Lax pair has been suggested in [22] to construct the hierarchies of nonlocal counterparts of the nonlinear equations admitting the compatibility condition representation. In this way, the hierarchies of the ILW_n equations, modified ILW_n equations and nonlocal Toda lattice have been obtained [22, 23]. Nonlocal bilinear equations for the latter system, which have been proposed in [24], should be modified to derive the multi-soliton solutions by applying the ordinary procedure of perturbation theory to a constant solution [25]. It is also inconvenient in the framework of the bilinear approach to determine a set of exponents, whose mutual products do not appear in the expansion for multi-soliton solutions, and, consequently, to describe soliton solutions.

In this paper, we extend the DT technique to nonlocal partners of Lax pairs and associated nonlocal nonlinear equations. The dual Lax pair with a quantized spectral parameter is introduced in section 2. We find the connections between the spaces of solutions of direct and dual Lax pairs. The reduction constraint on the Lax pair coefficients that leads to the nonlocal A_{n-1} Toda lattice ($n \in \mathbf{N}$) is discussed in section 3. This nonlinear system is written here in bilinear form, which is suitable for exploiting the usual perturbation theory and whose one-soliton expansion, as shown in the next section, has infinitely many exponential terms. In section 4, we present the theorem establishing the covariance of the Lax pairs with a quantized spectral parameter with respect to the DT of direct pairs. For the case of the Toda lattice, we formulate sufficient conditions to keep the reduction constraint on the Lax pair coefficients while performing this transformation. Similarly we construct the DT of a dual pair preserving the reduction, whose product with the DT of the direct pair yields the formulae of binary DT. Iterations of these transformations on zero background give multi-soliton solutions that depend on infinitely many arbitrary parameters. The formulae of infinitesimal DT and the expansions in series of the Lax pair solutions in the neighbourhood of singular points are presented in section 5. In this section, we give a simple way to produce the infinite hierarchies of the infinitesimal symmetries and conservation laws of the nonlocal A_{n-1} Toda lattice.

2. Lax pairs with a quantized spectral parameter

Let us consider the overdetermined linear system

$$\begin{cases} \Psi_x = -JT\Psi\Lambda + U\Psi \\ \Psi_t = AT^{-1}\Psi\Lambda^{-1} \end{cases} \tag{1}$$

for the matrix $n \times n$ function $\Psi \equiv \Psi(x, t, \Lambda)$. Here, $\Lambda = \mathrm{diag}(\lambda_j)$ is the constant matrix, T is the translation operator, $T = \exp(h\partial_x)$ (where h is a constant), and matrices J, U and A are independent of Λ. This system coincides with usual Lax pair studied in the framework of the IST method [6] if $h = 0$. The matrix Λ in this case is called the matrix spectral parameter. The joint action of operator T and matrix Λ on solution Ψ can be regarded as a notion of the quantized spectral parameter [22]. We refer to equations (1) as the direct Lax pair in the following. The compatibility condition $\Psi_{xt} = \Psi_{tx}$ of the direct Lax pair gives the system of nonlocal matrix equations:

$$\begin{cases} J_t = 0 \\ U_t = JTA - AT^{-1}J \\ A_x = UA - AT^{-1}U. \end{cases} \tag{2}$$

It is remarkable that equations (2) are also derived from the compatibility condition of the overdetermined linear system that is 'dual' to the Lax pair (1). This system (dual Lax pair) is written in following manner:

$$\begin{cases} \Xi_x = K(T^{-1}\Xi J) - \Xi U \\ \Xi_t = -K^{-1}(T\Xi A) \end{cases} \tag{3}$$

with $\Xi \equiv \Xi(x, t, K)$ and $K = \mathrm{diag}(æ_j)$ being the matrix solution and matrix spectral parameter of the dual system, respectively. The spaces of solutions of direct and dual Lax pairs are connected in the local case. Such a connection turns out to exist for the Lax pairs considered here. Namely, the matrix function

$$R(\Xi, \Psi) = K\int_x^{x+h}(T^{-1}\Xi J)\Psi\,\mathrm{d}x + \Xi\Psi$$

is independent of variables x and t if $K = \Lambda = \lambda E$ (λ is scalar spectral parameter). Another relation between the spaces of solutions is provided by the closure of a differential one-form

$$\mathrm{d}\omega(\Xi, \Psi) = \Xi J T\Psi\,\mathrm{d}x + K^{-1}(T\Xi A)\Psi\Lambda^{-1}\,\mathrm{d}t. \tag{4}$$

3. Nonlocal two-dimensional Toda lattice

Let matrix J be defined as given

$$J = \{\delta_{j,k-1}\} \tag{5}$$

(the indices are supposed hereafter to be equal on modulo n). System (2) is valid if we put

$$U = \sigma_x \qquad A = \exp(\sigma)J^{-1}\exp(-T^{-1}\sigma) \tag{6}$$

where

$$\sigma = \mathrm{diag}(\sigma_j) \tag{7}$$

and functions $\sigma_j \equiv \sigma_j(x, t)$ $(j = 1, \ldots, n)$ are the solutions of the nonlocal generalization of the two-dimensional periodic A_{n-1} Toda lattice [26]

$$\sigma_{j,xt} = \exp(T\sigma_{j+1} - \sigma_j) - \exp(\sigma_j - T^{-1}\sigma_{j-1}). \tag{8}$$

Changing variables

$$\sigma_j = \log\frac{T\tau_{j+1}}{\tau_j}$$

gives equations (8) in bilinear form

$$D_x D_t \tau_j \cdot \tau_j + 2\tau_j^2 = 2(T\tau_{j+1})T^{-1}\tau_{j-1} \tag{9}$$

(D_x and D_t denote the Hirota derivatives [25]). In the local case, these equations give the bilinear representation of the two-dimensional Toda lattice (compare with equation (4.3) in [24]).

The important feature of nonlocal direct and dual Lax pairs is that the matrix U of general form cannot be led to the algebra $SL(n)$ case by means of the gauge transformation [6]. However, the coefficients of linear systems (1) and (3) admit a new reduction constraint which is in keeping with system (8)

$$\sigma = \rho E \tag{10}$$

where a single dependent variable ρ solves the equation (A_0 Toda lattice)

$$\rho_{xt} = (T - 1)\exp(\rho - T^{-1}\rho). \tag{11}$$

It should be stressed that this equation, whose local analogue is trivial, appears as the compatibility condition of the overdetermined linear systems of the arbitrary matrix dimension. Making the dependent variable transformation in equation (11)

$$\rho = (T - 1) \log \tau$$

one obtains the bilinear equation for τ

$$D_x D_t \tau \cdot \tau + 2\tau^2 = 2(T\tau)T^{-1}\tau. \tag{12}$$

Equation (11) was derived in [27] as the continuous limit of the lattice equations describing the transfer of energy of plasma oscillations. The nonlocal equation that can be written in the form of equation (11), with operator T defined in a different manner, was considered in [21].

The soliton solutions of equations (9) and (11) can be constructed applying the usual procedure of the Hirota method [25]. However, the explicit form of the one-soliton solution is not obvious in this approach. To generate the hierarchy of solutions of the nonlocal Toda lattice we develop another method here.

4. Darboux transformation technique and solitons

The underlying property of the DT technique is the existence of the kernel of transformations of the Lax pair solutions for some value of the spectral parameter. This property is exploited in this paper to generalize the technique considered for the nonlocal equations. For the sake of convenience we introduce notation

$$\Omega(\Xi, \Psi) = \int_{(x_0,t_0)}^{(x,t)} \mathrm{d}\omega\,(\Xi, \Psi) + C(\Xi, \Psi)$$

where it is supposed that the constant matrix $C(\Xi, \Psi)$ can be determined from equation

$$KC(\Xi, \Psi) - C(\Xi, \Psi)\Lambda = R(\Xi, \Psi)|_{(x_0,t_0)}.$$

Theorem. *Let Φ be a solution of direct Lax pair (1) with matrix spectral parameter M. Then matrix*

$$\tilde{\Psi} = \Psi_x - \Phi_x\Phi^{-1}\Psi \tag{13}$$

and, if there exists an appropriate matrix $C(\Xi, \Phi)$, matrix

$$\tilde{\Xi} = \Omega(\Xi, \Phi)(T\Phi^{-1})J^{-1} \tag{14}$$

are solutions respectively of direct and dual Lax pairs

$$\begin{cases} \tilde{\Psi}_x = -JT\tilde{\Psi}\Lambda + \tilde{U}\tilde{\Psi} \\ \tilde{\Psi}_t = \tilde{A}T^{-1}\tilde{\Psi}\Lambda^{-1} \end{cases} \qquad \begin{cases} \tilde{\Xi}_x = K(T^{-1}\tilde{\Xi}J) - \tilde{\Xi}\tilde{U} \\ \tilde{\Xi}_t = -K^{-1}(T\tilde{\Xi}\tilde{A}) \end{cases}$$

whose coefficients are

$$\tilde{U} = U + J_x J^{-1} + J(T\Phi_x\Phi^{-1})J^{-1} - \Phi_x\Phi^{-1} \tag{15}$$

$$\tilde{A} = J(T\Phi)M\Phi^{-1}A(T^{-1}\Phi)M^{-1}\Phi^{-1}(T^{-1}J^{-1}). \tag{16}$$

The direct Lax pair is covariant with respect to transformation $\{\Psi, U, A\} \to \{\tilde{\Psi}, \tilde{U}, \tilde{A}\}$ owing to the existence of the kernel: $\tilde{\Psi} \equiv 0$ if $\Psi = \Phi$. The covariance of the dual Lax pair is proven by applying identity

$$K(T^{-1}\Omega(\Xi, \Phi)) = \Omega(\Xi, \Phi)M + \Xi\Phi.$$

It follows from the compatibility conditions of transformed direct or dual Lax pairs that their coefficients J, $\tilde{U}$ and $\tilde{A}$ are new solutions of system (2). The transformation (13)–(16) carried out with solution Φ of equation (1) is called the DT of the direct pair. The reduction constraint (6) on the Lax pair coefficients is kept while performing this DT by imposing the following conditions on solution Φ and its spectral parameter M:

$$\Phi = \mathrm{diag}(\varphi_j)B \tag{17}$$

$$\begin{aligned} B_{jk} &= \exp(2\pi \mathrm{i}(j-1)(k-1)/n) \qquad (j,k = 1,\dots,n) \\ M &= \mu\ \mathrm{diag}(\exp(2\pi \mathrm{i}(j-1)/n)) \end{aligned} \tag{18}$$

where $\varphi \equiv \varphi(x,t,\mu) = (\varphi_1,\dots,\varphi_n)^T$ is a vector solution of equation (1) with the scalar spectral parameter μ. If $\psi = (\psi_1,\dots,\psi_n)^T$ and $\xi = (\xi_1,\dots,\xi_n)$ are the vector solutions of direct and dual Lax pairs with scalar spectral parameters λ and æ respectively, then equations (13) and (14) under conditions (17) and (18) give the following expressions for the components of transformed vector solutions:

$$\tilde{\psi}_j = \psi_{j,x} - \frac{\varphi_{j,x}}{\varphi_j}\psi_j \tag{19}$$

$$\tilde{\xi}_j = \frac{\Omega_{j+1}(\xi,\varphi)}{T\varphi_{j+1}} \tag{20}$$

$(j = 1,\dots,n)$ with

$$\Omega_j(\xi,\varphi) = \int_{(x_0,t_0)}^{(x,t)} \xi_{j-1}T\varphi_j\,\mathrm{d}x + \exp(T\sigma_j - \sigma_{j-1})\frac{(T\xi_j)\varphi_{j-1}}{æ\mu}\,\mathrm{d}t + \left.\frac{\rho_j(\xi,\varphi)}{æ^n - \mu^n}\right|_{(x_0,t_0)}$$

$$\rho_j(\xi,\varphi) = \sum_{k=1}^{n} æ^{k-1}\mu^{n-k}\left(æ\int_x^{x+h}(T^{-1}\xi_{j-k-1})\varphi_{j-k}\,\mathrm{d}x + \xi_{j-k}\varphi_{j-k}\right).$$

The transformation of solutions of the nonlocal Toda lattice (8), which corresponds to transformations (19) and (20) of the solutions of associated Lax pairs, has the form

$$\tilde{\sigma}_j = \sigma_j + \log\frac{T\varphi_{j+1}}{\varphi_j} \qquad (j = 1,\dots,n). \tag{21}$$

Similarly one can construct the DT formulae of the dual pair, using the matrix solution of equation (3). It can be shown that the transformations of direct and dual pairs commute. The conditions analogous to equations (17) and (18) are imposed on the matrix solution and its spectral parameter for reduction constraint (6) to be inherited while carrying out the DT of the dual pair. The corresponding formulae for transformations of Lax pair solutions and those of the nonlocal Toda lattice are

$$\tilde{\psi}_j = T^{-1}\frac{\Omega_j(\chi,\psi)}{\chi_{j-1}} \tag{22}$$

$$\tilde{\xi}_j = \xi_{j,x} - \frac{\chi_{j,x}}{\chi_j}\xi_j \tag{23}$$

$$\tilde{\sigma}_j = \sigma_j + \log\frac{\chi_j}{T^{-1}\chi_{j-1}} \tag{24}$$

$(j = 1,\dots,n)$, where $\chi = (\chi_1,\dots,\chi_n)$ is the vector solution of the dual Lax pair with scalar spectral parameter ν. The product of transformations (19)–(21) and (22)–(24) yields the formulae of the so-called binary DT

$$\tilde{\psi}_j = \psi_j - \frac{T^{-1}\Omega_j(\chi,\psi)}{T^{-1}\Omega_j(\chi,\varphi)}\varphi_j \tag{25}$$

$$\tilde{\xi}_j = \xi_j - \frac{\Omega_{j+1}(\xi,\varphi)}{\Omega_{j+1}(\chi,\varphi)}\chi_j \tag{26}$$

$$\tilde{\sigma}_j = \sigma_j + \log\frac{\Omega_{j+1}(\chi,\varphi)}{T^{-1}\Omega_j(\chi,\varphi)} \tag{27}$$

$(j = 1, \ldots, n)$.

To keep reduction constraint (10) we have to impose additional conditions on solutions of Lax pairs used in performing DTs. For example, if we put $\varphi_j = \alpha^j\vartheta$ in transformation (19)–(21), where $\alpha = \exp(2\pi \mathrm{i}k/n)$ $(k = 0, \ldots, n-1)$ and ϑ solves system

$$\begin{cases} \vartheta_x = -\alpha\mu T\vartheta + \rho_x\vartheta \\ \vartheta_t = \frac{\exp(\rho - T^{-1}\rho)}{\alpha\mu}T^{-1}\vartheta \end{cases}$$

then the transformed solution of equation (11) is

$$\tilde{\rho} = \rho + (T-1)\log\vartheta.$$

Let us consider the zero background $(\sigma = 0)$. Vectors φ and χ, which were exploited in constructing the DTs, satisfy nonlocal linear systems

$$\begin{cases} \varphi_x = -\mu J T\varphi \\ \varphi_t = \mu^{-1}J^{-1}T^{-1}\varphi \end{cases} \tag{28}$$

$$\begin{cases} \chi_x = \nu T^{-1}\chi J \\ \chi_t = -\nu^{-1}T\chi J^{-1}. \end{cases} \tag{29}$$

These systems have solutions depending on infinitely many parameters of the following form:

$$\varphi_j = \sum_{k=0}^{n-1}\sum_{m_k} c_k^{(m_k)} \exp(2\pi\mathrm{i}(j-1)k/n)\exp\left(p_k^{(m_k)}x - t\big/p_k^{(m_k)}\right) \tag{30}$$

$$\chi_j = \sum_{k=0}^{n-1}\sum_{n_k} d_k^{(n_k)} \exp(-2\pi\mathrm{i}(j-1)k/n)\exp\left(q_k^{(n_k)}x - t\big/q_k^{(n_k)}\right) \tag{31}$$

$(j = 1, \ldots, n)$. Here $p_k^{(m_k)}$ and $q_k^{(n_k)}$ satisfy equations

$$p_k^{(m_k)} + \mu\exp(2\pi\mathrm{i}k/n)\exp\left(p_k^{(m_k)}h\right) = 0$$

$$q_k^{(n_k)} - \nu\exp(2\pi\mathrm{i}k/n)\exp\left(-q_k^{(n_k)}h\right) = 0$$

where $(k = 0, \ldots, n-1)$, $c_k^{(m_k)}$ and $d_k^{(n_k)}$ are arbitrary constants. Carrying out DT according to equations (19)–(21) (or equations (22)–(24)) gives the one-soliton solutions, which depend on infinitely many free parameters. This means that, in general, the set τ_j $(j = 1, \ldots, n)$ of the one-soliton τ-functions of equations (9) and (12) contain infinitely many exponential terms. The products of these exponents do not appear in the expression of the multi-soliton solutions of the bilinear equations. One-soliton solutions in the nonlocal case form two families to be produced by DTs of direct and dual pairs respectively, while in the local case the transformation (22)–(24) can be obtained iterating the transformation (19)–(21).

For $n = 2$, the components of the solution of system (28) are represented in the following manner:

$$\varphi_j = \sum_{p_+} c_{p_+} \exp(p_+ x - t/p_+) + (-1)^{j-1} \sum_{p_-} c_{p_-} \exp(p_- x - t/p_-)$$

$(j = 1, 2)$, where $p_\pm$ satisfy equations

$$p \pm \mu \exp(ph) = 0$$

and the summations are supposed over all solutions of the latter equations. The substitution of the vector solution φ, such that $\varphi_1 = \pm\varphi_2$, into equation (21) leads to the solutions of the A_0 Toda lattice (11). For real h, the simplest real nonsingular solution is

$$\rho = (T - 1)\log\vartheta$$

where

$$\vartheta = c_1 \exp(p_1 x - t/p_1) + c_2 \exp(p_2 x - t/p_2).$$

c_1, c_2 and μ are real constants, and $c_1 c_2 > 0$, p_1 and p_2 are real solutions of the equation

$$p = \mu \exp(ph).$$

The iterations of transformations (19)–(27) allow us to construct the infinite hierarchies of solutions of equations (8) and the corresponding solutions of Lax pairs (1) and (3). The final expressions for the transformed quantities are brought to the determinant form. Taking into account

$$\lambda\rho_{j+1}(\xi, \psi) = æ\rho_j(\xi, \psi) + (\lambda^n - æ^n)\left(æ\int_x^{x+h} (T^{-1}\xi_{j-1})\psi_j \,\mathrm{d}x + \xi_j\psi_j\right)$$

$$æT^{-1}\Omega_j(\xi, \psi) = \lambda\Omega_{j+1}(\xi, \psi) + \xi_j\psi_j$$

we obtain

$$\rho_j(\tilde{\xi}, \tilde{\psi}) = (æ^n - \lambda^n)\frac{T\psi_j}{T\varphi_j}\Omega_j(\xi, \varphi) - \rho_j(\xi, \psi)$$

$$\Omega_j(\tilde{\xi}, \tilde{\psi}) = \frac{T\psi_j}{T\varphi_j}\Omega_j(\xi, \varphi) - \Omega_j(\xi, \psi)$$

where $\tilde{\psi}$ and $\tilde{\xi}$ are defined according to equations (19) and (20). The formulae of the Nth iteration of the DT of the direct pair are written using these equations as given

$$\tilde{\psi}_j = \frac{\Delta^{(j)}[N+1]}{\Delta^{(j)}[N]}$$

$$\tilde{\xi}_j = \begin{vmatrix} T\varphi^{(1)}_{j+1} & T\varphi^{(1)}_{j+1,x} & \dots & T\varphi^{(1)}_{j+1,(N-2)x} & \Omega_{j+1}(\xi, \varphi^{(1)}) \\ T\varphi^{(2)}_{j+1} & T\varphi^{(2)}_{j+1,x} & \dots & T\varphi^{(2)}_{j+1,(N-2)x} & \Omega_{j+1}(\xi, \varphi^{(2)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T\varphi^{(N)}_{j+1} & T\varphi^{(N)}_{j+1,x} & \dots & T\varphi^{(N)}_{j+1,(N-2)x} & \Omega_{j+1}(\xi, \varphi^{(N)}) \end{vmatrix} \Bigg/ T\Delta^{(j+1)}[N]$$

$$\tilde{\sigma}_j = \sigma_j + \log\frac{T\Delta^{(j+1)}[N]}{\Delta^{(j)}[N]}$$

where

$$\Delta^{(j)}[N] = \det\left[\varphi^{(k)}_{j,(m-1)x}\right].$$

$\varphi^{(k)}_j$ are the components of the vector solutions of equation (1) with scalar spectral parameter $\mu^{(k)}$, $\varphi^{(N+1)} = \psi$ $(j = 1, \dots, n; k, m = 1, \dots, N)$. The reduction constraint (10) is inherited under the iterations by supposing $N = ln\,(l = 1, 2, \ldots)$, $\varphi^{(ml+k)} = \left(\varphi^{(k)}_{m+1}, \ldots, \varphi^{(k)}_{m+n}\right)^T$, $\mu^{(ml+k)} = \mu^{(k)}$ $(m = 0, \ldots, n-1;\ k = 1, \ldots, l)$.

The iterations of the transformations presented in this section on zero background give the multi-soliton solutions of the nonlocal A_{n-1} Toda lattice that depend on infinitely many arbitrary parameters. The interaction of the one-soliton components in the multi-soliton solution causes the shifts of the parameters of the one-soliton solutions.

5. Infinitesimal symmetries and conservation laws

Taking the limit $\nu \to \mu$ in equations (25)–(27) one finds solutions of the linearizations of the direct and dual Lax pairs and the nonlocal Toda lattice

$$\delta\psi_j = \mu(T^{-1}\Omega_j(\chi, \psi))\varphi_j \tag{32}$$

$$\delta\xi_j = \mu\Omega_{j+1}(\xi, \varphi)\chi_j \tag{33}$$

$$\delta\sigma_j = \chi_j\varphi_j \tag{34}$$

$(j = 1, \ldots, n)$. These formulae establish infinitesimal DT $\psi \to \psi + \varepsilon\delta\psi$, $\xi \to \xi + \varepsilon\delta\xi$, $\sigma \to \sigma + \varepsilon\delta\sigma$ $(\varepsilon = \nu - \mu)$. The closure of the differential one-form (4) immediately yields

$$(\Xi J T\Psi)_t = (K^{-1}(T\Xi A)\Psi\Lambda^{-1})_x.$$

In terms of the vector solutions φ and χ of the Lax pairs with scalar spectral parameter μ, this identity reads as

$$\left(\sum_{j=1}^{n}\chi_{j-1}T\varphi_j\right)_t = \mu^{-2}\left(\sum_{j=1}^{n}(T\chi_{j+1})\varphi_j\,\mathrm{e}^{T\sigma_j-\sigma_{j-1}}\right)_x$$

or, equivalently,

$$\mathcal{T}_t + X_x = 0 \tag{35}$$

where we use notations

$$\mathcal{T} = \sum_{j=1}^{n}\chi_{j-1}T\varphi_j \tag{36}$$

$$X = -\mu^{-2}\sum_{j=1}^{n}\left(T\chi_{j+1}\right)\varphi_j\,\mathrm{e}^{T\sigma_j-\sigma_{j-1}}. \tag{37}$$

The hierarchies of infinitesimal symmetries and conservation laws of the nonlocal generalization of the two-dimensional periodic A_{n-1} Toda lattice are obtained by substituting into equations (34)–(37) the expansions of Lax pair solutions in the neighbourhood of singular points on the spectral parameter plane. The components of the vector solutions φ and χ of the direct and dual Lax pairs (1) and (3), whose coefficients are defined by equations (5)–(7), are represented in the neighbourhood of point $\mu = \infty$ in the following manner:

$$\varphi_j = \left(1 + \sum_{k=1}^{\infty}\frac{A_j^{(k)}}{\Omega^k}\right)\left(\frac{\Omega}{\mu}\right)^{j-1}\mathrm{e}^{\Omega(j-1)h-\Omega x} \tag{38}$$

$$\chi_j = \left(1 + \sum_{k=1}^{\infty}\frac{B_j^{(k)}}{\Omega^k}\right)\left(\frac{\mu}{\Omega}\right)^{j-1}\mathrm{e}^{-\Omega(j-1)h+\Omega x} \tag{39}$$

$(j = 1, \ldots, n)$, where Ω solves the equation

$$\Omega^n\exp(n\Omega h) = \mu^n$$

and coefficients $A_j^{(k)}$ and $B_j^{(k)}$ ($k \in \mathbf{N}$) satisfy equations

$$\begin{cases} -A_j^{(k)} + A_{j,x}^{(k-1)} = -TA_{j+1}^{(k)} + \sigma_{j,x}A_j^{(k-1)} \\ A_{j,t}^{(k)} = \mathrm{e}^{\sigma_j - T^{-1}\sigma_{j-1}}T^{-1}A_{j-1}^{(k-1)} \end{cases}$$

$$\begin{cases} B_j^{(k)} + B_{j,x}^{(k-1)} = T^{-1}B_{j-1}^{(k)} - \sigma_{j,x}B_j^{(k-1)} \\ B_{j,t}^{(k)} = -\mathrm{e}^{T\sigma_{j+1}-\sigma_j}TB_{j+1}^{(k-1)}. \end{cases}$$

In the neighbourhood of singular point $\mu = 0$ we have

$$\varphi_j = \sum_{k=0}^{\infty} C_j^{(k)}\alpha^{1-j-k}\mu^k\,\mathrm{e}^{\alpha t/\mu} \tag{40}$$

$$\chi_j = \sum_{k=0}^{\infty} D_j^{(k)}\alpha^{j-k-1}\mu^k\,\mathrm{e}^{-\alpha t/\mu} \tag{41}$$

($j = 1, \ldots, n$). Here α is a root of the equation

$$\alpha^n = 1$$

and $C_j^{(k)}$ and $D_j^{(k)}$ are solutions of the systems

$$\begin{cases} C_{j,x}^{(k+1)} = -TC_{j+1}^{(k)} + \sigma_{j,x}C_j^{(k+1)} \\ C_{j,t}^{(k)} + C_j^{(k+1)} = \mathrm{e}^{\sigma_j - T^{-1}\sigma_{j-1}}T^{-1}C_{j-1}^{(k+1)} \end{cases}$$

$$\begin{cases} D_{j,x}^{(k+1)} = T^{-1}D_{j-1}^{(k)} - \sigma_{j,x}D_j^{(k+1)} \\ D_{j,t}^{(k)} - D_j^{(k+1)} = -\mathrm{e}^{T\sigma_{j+1}-\sigma_j}TD_{j+1}^{(k+1)}. \end{cases}$$

We can check by straightforward calculations that the systems determining coefficients $A_j^{(k)}$, $B_j^{(k)}$, $C_j^{(k)}$ and $D_j^{(k)}$ are compatible. The formulae for the expansions in series in the neighbourhood of singular points of the Lax pair solutions in the nonlocal case seem to be new. Note that, for $\mu = \infty$, we have infinitely many types of asymptotic behaviour. It follows after substituting expansions (38), (39) and (40), (41) in equations (34), (36) and (37) that the solution of the linearization of the nonlocal Toda lattice, conserved densities and currents admit the representation

$$\delta\sigma_j = \sum_{k=0}^{\infty}\delta\sigma_j^{(k,\infty)}\Omega^{-k} \qquad \delta\sigma_j = \sum_{k=0}^{\infty}\delta\sigma_j^{(k,0)}\mu^k$$

$$T = \sum_{k=0}^{\infty}\sum_{j=1}^{n}T_j^{(k,\infty)}\Omega^{-k} \qquad T = \sum_{k=0}^{\infty}\sum_{j=1}^{n}T_j^{(k,0)}\mu^k$$

$$X = \sum_{k=0}^{\infty}\sum_{j=1}^{n}X_j^{(k,\infty)}\Omega^{-k} \qquad X = \sum_{k=0}^{\infty}\sum_{j=1}^{n}X_j^{(k,0)}\mu^k$$

whose coefficients form the infinite hierarchies of infinitesimal symmetries and conservation laws. The first few nontrivial coefficients are

$$\delta\sigma_j^{(1,0)} = \sigma_{j,t} \qquad \delta\sigma_j^{(1,\infty)} = \sigma_{j,x} \qquad \delta\sigma_j^{(2,\infty)} = \sigma_{j,x}^2 + \int^t \left(\mathrm{e}^{T\sigma_{j+1}-\sigma_j} + \mathrm{e}^{\sigma_j - T^{-1}\sigma_{j-1}}\right)_x \mathrm{d}t.$$

$$T_j^{(2,\infty)} = \int^t \left(\mathrm{e}^{T\sigma_j - \sigma_{j-1}}\int^t \left(\mathrm{e}^{T^2\sigma_{j+1}-T\sigma_j} + \mathrm{e}^{\sigma_{j-1}-T^{-1}\sigma_{j-1}}\right)\mathrm{d}t\right)\mathrm{d}t - \left(\int^t \mathrm{e}^{T\sigma_j-\sigma_{j-1}}\,\mathrm{d}t\right)^2,$$

$$X_j^{(2,\infty)} = -\mathrm{e}^{T\sigma_j-\sigma_{j-1}}.$$

6. Conclusion

In this paper, the multi-soliton solutions of the nonlocal partner of the two-dimensional Toda lattice have been obtained. Unlike in the local case, these solutions depend on infinitely many free parameters. This feature of multi-soliton solution follows in the framework of the Darboux transformation technique from the analogous property of the solutions of the Lax pairs of the nonlocal Toda lattice. We have also presented the formulae of the expansions in series on the spectral parameter powers of the solutions of the Lax pairs. These expansions have been used to construct the hierarchies of infinitesimal symmetries and conservation laws of the nonlocal two-dimensional Toda lattice.

Acknowledgments

I thank Dr Heinz Steudel for useful discussions and hospitality. I also thank Gottlieb Daimler and Karl Benz-Stiftung for financial support.

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